Optical Waveguide Theory (I)

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Course overview

Optical waveguide theory

A  Photonics / integrated optics; theory, motto; phenomena, introductory examples.
B  Brush up on mathematical tools.
C  Maxwell equations, different formulations, interfaces, energy and power flow.
D  Classes of simulation tasks: scattering problems, mode analysis, resonance problems.
E  Normal modes of dielectric optical waveguides, mode interference.
F  Examples for dielectric optical waveguides.
G  Waveguide discontinuities & circuits, scattering matrices, reciprocal circuits.
H  Bent optical waveguides; whispering gallery resonances; circular microresonators.
   I  Coupled mode theory, perturbation theory.
   ● Hybrid analytical / numerical coupled mode theory.
J  A touch of photonic crystals; a touch of plasmonics.
   ● Oblique semi-guided waves: 2-D integrated optics.
   ● Summary, concluding remarks.
Perturbations of single modes

$\sim \exp(i\omega t)$ (FD)

$\hat{\epsilon}$

$\lambda, \hat{\epsilon}(x, y)$

$\beta, \bar{E}, \bar{H}$
Perturbations of single modes

\[ \sim \exp(i\omega t) \text{ (FD)} \]

\[ \hat{\epsilon} + \delta\hat{\epsilon} \]

\[ \lambda, \ \hat{\epsilon}(x, y) + \delta\hat{\epsilon}(x, y) \]

\[ \beta + \delta\beta, \ \bar{E} + \delta\bar{E}, \ \bar{H} + \delta\bar{H} \]

\[ \lambda, \ \hat{\epsilon}(x, y) \]

\[ \beta, \ \bar{E}, \ \bar{H} \]
Perturbations of single modes

\[ \sim \exp(i\omega t) \] (FD)

\[ \hat{\epsilon} \]

\[ \lambda, \; \hat{\epsilon}(x, y) \]

\[ \beta, \; \bar{E}, \; \bar{H} \]

\[ \epsilon + \delta \hat{\epsilon} \]

\[ \lambda, \; \hat{\epsilon}(x, y) + \delta \hat{\epsilon}(x, y) \]

\[ \beta + \delta \beta, \; \bar{E} + \delta \bar{E}, \; \bar{H} + \delta \bar{H} \]
A functional for guided modes of 3-D dielectric waveguides

• \((E H) (x, y, z) = (\bar{E} \bar{H}) (x, y) e^{-i \beta z}, \quad \beta \in \mathbb{R}, \quad \bar{E}, \bar{H} \to 0 \text{ for } x, y \to \pm \infty.\)

• \((C + i \beta R)\bar{E} = -i \omega \mu_0 \bar{H}, \quad (C + i \beta R)\bar{H} = i \omega \epsilon_0 \hat{\epsilon} \bar{E},\)

\(R = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & \partial_y \\ 0 & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix}.\)

• \(B_\hat{\epsilon}(\bar{E}, \bar{H}) := \frac{\omega \epsilon_0 \langle \bar{E}, \hat{\epsilon} \bar{E} \rangle + \omega \mu_0 \langle \bar{H}, \bar{H} \rangle + i \langle \bar{E}, C \bar{H} \rangle - i \langle \bar{H}, C \bar{E} \rangle}{\langle \bar{E}, R \bar{H} \rangle - \langle \bar{H}, R \bar{E} \rangle},\)

\(\langle \bar{F}, \bar{G} \rangle = \int \int \bar{F}^* \cdot \bar{G} \, dx \, dy.\)

\(B_\hat{\epsilon}(\bar{E}, \bar{H}) = \beta \quad (\ast), \quad \frac{d}{ds} B_\hat{\epsilon}(\bar{E} + s \bar{F}, \bar{H} + s \bar{G}) \bigg|_{s=0} = 0 \quad (\ast \ast)\)

at valid mode fields \(\bar{E}, \bar{H},\) for arbitrary \(\bar{F}, \bar{G}.\)

(\ast): “arbitrary” \(\hat{\epsilon}_.\)

(\ast \ast): Hermitian \(\hat{\epsilon}_.\)
Perturbations of single modes

• Available: Mode $\beta, \bar{E}, \bar{H}$ for parameters $\lambda, \hat{\epsilon}$; $\mathcal{B}_{\hat{\epsilon}}(\bar{E}, \bar{H}) = \beta$, $\mathcal{B}_{\hat{\epsilon}}$ stationary at $\bar{E}, \bar{H}$.

$B_{\hat{\epsilon}}(\bar{E}, \bar{H}) = \beta$, \(\hat{\epsilon} = \hat{\epsilon}^\dagger\)

• Investigate parameters $\lambda, \hat{\epsilon} + \delta \hat{\epsilon}$, for a “small” change $\delta \hat{\epsilon}$:

$\mathcal{B}_{\hat{\epsilon} + \delta \hat{\epsilon}}(\bar{E} + \delta \bar{E}, \bar{H} + \delta \bar{H}) = \beta + \delta \beta$

\[ \mathcal{B}_{\hat{\epsilon}}(\bar{E} + \delta \bar{E}, \bar{H} + \delta \bar{H}) \approx \mathcal{B}_{\hat{\epsilon}}(\bar{E}, \bar{H}) = \beta \]

\[ \delta(\cdot) \delta(\cdot) \]

\[ \delta \beta = \frac{\omega \epsilon_0 \langle \bar{E}, \delta \hat{\epsilon} \bar{E} \rangle}{\langle \bar{E}, \mathcal{R} \bar{H} \rangle - \langle \bar{H}, \mathcal{R} \bar{E} \rangle} \], or \[ \delta \beta = \frac{\omega \epsilon_0}{2 \Re} \iint (\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x) \, dx \, dy \]

(Valid for small perturbations: The original mode profiles are good approximations of the true fields in the modified structure.)
Small uniform change in refractive index

- \( n \rightarrow n + \delta n \) on \( \Box \), \( n, \delta n \) constant on \( \Box \)

- \( \beta \rightarrow \beta + \delta \beta, \quad \delta \beta = \frac{\omega \epsilon_0 n \iint_\Box |\vec{E}|^2 \, dx \, dy}{\text{Re} \iint_\Box (\bar{E}_x \bar{H}_y - \bar{E}_y \bar{H}_x) \, dx \, dy} \delta n \).

(Plausible: \( \delta \epsilon = 2n \delta n \).
(Plausible: \( \delta \beta \sim \delta n, \ \delta \beta \sim |\vec{E}|^2 \).)
Small attenuation

\[
\hat{\epsilon} + \delta\hat{\epsilon}
\]

- \( n \rightarrow n - i n'' \) on \( \Box \), \( n, n'' \) constant on \( \Box \), \( n, n'' \in \mathbb{R} \)

\[
\beta \rightarrow \beta + \delta\beta, \quad \delta\beta = \frac{-i\omega\epsilon_0 n \iint_{\Box} |\vec{E}|^2 \, dx \, dy}{\text{Re} \iint (\vec{E}^*_x \vec{H}_y - \vec{E}^*_y \vec{H}_x) \, dx \, dy} n''.
\]

(\( \delta\epsilon = -i2nn'' \).)

(Different attenuation for each mode.)

(Damping, power, plane wave: \( \sim \exp(-2kn''z) \), mode: \( \not\sim \exp(-2kn''z) \).)
Small anisotropy

\[ \epsilon \hat{1} \longrightarrow \epsilon \hat{1} + \delta \hat{\epsilon} \text{ on } \square, \quad \epsilon, \delta \hat{\epsilon} \text{ constant on } \square \]

\[ \beta \longrightarrow \beta + \delta \beta, \quad \delta \beta = \frac{\omega \epsilon_0}{2 \text{ Re}} \int \int \bar{E}^* \cdot \delta \hat{\epsilon} \bar{E} \, dx \, dy \]

\( \int \int (\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x) \, dx \, dy \)

(Phase shifts due to anisotropic contributions to the permittivity.)

(Polarization coupling might occur for modes with “close” propagation constants CMT.)
Small displacements of dielectric interfaces

Interface displacement $\xrightarrow{\sim}$ Locally strong thin layer perturbation.
Field discontinuity $\xrightarrow{\sim}$ Previous expressions are not directly applicable.

- $\epsilon^- \neq \epsilon^+$, shift of interface $x_b \rightarrow x_b + \delta x$.

- Reposition discontinuity in field: $E_x \rightarrow E_x + \delta E_x$,

$$\delta E_x(x, y) = \begin{cases} 
\frac{\epsilon^+ - \epsilon^-}{\epsilon^-} E_x(x, y), & \text{for } x_b < x < x_b + \delta x, \\
0, & \text{otherwise.}
\end{cases}$$

- Use functional with locally modified field

\[ \ldots \text{(omitted)} \ldots \]
Small displacements of dielectric interfaces

Displacement of the interface at $x_b$ between $y_0$ and $y_1$ by $\delta x$:

$$\beta \rightarrow \beta + \delta \beta,$$

$$\delta \beta = \frac{\omega \varepsilon_0}{2} \frac{(\varepsilon^- - \varepsilon^+) \int_{y_0}^{y_1} \left( \frac{1}{\varepsilon^- \varepsilon^+} |\varepsilon \bar{E}_x|^2 + |\bar{E}_y|^2 + |\bar{E}_z|^2 \right) (x_b, y) \, dy}{\text{Re} \int \int (\bar{E}_x^* \bar{H}_y - \bar{E}_y^* \bar{H}_x) \, dx \, dy} \, \delta x.$$
Small displacements of dielectric interfaces

- Displacement of the interface at $y_b$ between $x_0$ and $x_1$ by $\delta y$:

$$\beta \rightarrow \beta + \delta \beta,$$

$$\delta \beta = \frac{\omega \epsilon_0}{2} \sum_{x_0}^{x_1} \left( |E_x|^2 + \frac{1}{\epsilon^- - \epsilon^+} |E_y|^2 + |E_z|^2 \right) (x, y_b) \, dx$$

$$\delta \beta = \frac{\omega \epsilon_0}{2} \frac{\epsilon^- - \epsilon^+}{\epsilon^- - \epsilon^+} \int_{x_0}^{x_1} \left( |E_x|^2 + \frac{1}{\epsilon^- - \epsilon^+} |E_y|^2 + |E_z|^2 \right) (x, y_b) \, dx$$

$$\delta \beta = \frac{\omega \epsilon_0}{2} \frac{\epsilon^- - \epsilon^+}{\epsilon^- - \epsilon^+} \left( \operatorname{Re} \int \int (E_x^* H_y - E_y^* H_x) \, dx \, dy \right) \delta y.$$
Perturbations of single modes

\[ \lambda, \ \hat{e}(x, y) \]
\[ \beta, \ \bar{E}, \ \bar{H} \]

\[ \lambda, \ \hat{e}(x, y) + \delta \hat{e}(x, y) \]
\[ \beta + \delta \beta, \ \bar{E}, \ \bar{H} \]

- View \( \frac{\delta \beta}{\delta p} \) as \( \frac{\partial \beta}{\partial p} \): slope of the dispersion curves \( \beta \) vs. \( p \).
- Depending on the parametrization, change of a parameter value might require several perturbations.
- First order theory: In case of multiple perturbations, add the effects of the individual expressions.
- Estimation of fabrication tolerances: The phase shifts \( \delta \beta \) enter into respective scattering matrix models.
- Wavelength shifts . . . ?
Small shift of frequency or vacuum wavelength

\[ \beta(\omega) = \mathcal{B}(\omega; \vec{E}(\omega), \vec{H}(\omega)) \]

\[ \frac{\partial \beta}{\partial \omega} = \frac{\partial \mathcal{B}}{\partial \omega} \quad (\ast) + \frac{\partial}{\partial s} \mathcal{B}(\omega; \vec{E} + s \frac{\partial \vec{E}}{\partial \omega}, \vec{H}) \bigg|_{s=0} \quad (** \ast) \]

\[ + \frac{\partial}{\partial s} \mathcal{B}(\omega; \vec{E}, \vec{H} + s \frac{\partial \vec{H}}{\partial \omega}) \bigg|_{s=0} \quad (** \ast) \]

(Stationarity of \( \mathcal{B} \) at \( \vec{E}, \vec{H} \).)

\[ \frac{\partial \beta}{\partial \omega} = \int \int \left( \epsilon_0 \vec{E}^* \cdot \frac{\partial (\omega \hat{\epsilon})}{\partial \omega} \vec{E} + \mu_0 |\vec{H}|^2 \right) \, dx \, dy \]

\[ = \frac{\partial \mathcal{B}}{\partial \omega} \frac{2 \text{Re} \int \int \left( \vec{E}^* \vec{H}_y - \vec{E}_y^* \vec{H}_x \right) \, dx \, dy}{\int \int \left( \vec{E}^* \vec{H}_y - \vec{E}_y^* \vec{H}_x \right) \, dx \, dy} . \]
Small shift of frequency or vacuum wavelength

If dispersion can be neglected, \( \partial_\omega \hat{\epsilon} = 0 \):

\[
\frac{\partial \beta}{\partial \omega} = \frac{\int \int \left( \epsilon_0 \bar{E}^* \cdot \hat{\epsilon} \bar{E} + \mu_0 |\bar{H}|^2 \right) \, dx \, dy}{2 \Re \int \int (\bar{E}_x \bar{H}_y - \bar{E}_y \bar{H}_x) \, dx \, dy},
\]

\[
\frac{\partial \beta}{\partial \lambda} = -\frac{\pi c}{\lambda^2} \frac{\int \int \left( \epsilon_0 \bar{E}^* \cdot \hat{\epsilon} \bar{E} + \mu_0 |\bar{H}|^2 \right) \, dx \, dy}{\Re \int \int (\bar{E}_x \bar{H}_y - \bar{E}_y \bar{H}_x) \, dx \, dy}.
\]

\((\omega = 2\pi c / \lambda \quad \partial_\lambda \omega = -2\pi c / \lambda^2)\)

(Compare with expression based on homogeneity, H, 12.)
Coupled mode theory (CMT)

\[ \sim \exp(i\omega t) \] (FD)

\[ \left\{ \hat{\epsilon}_m; \beta_m, (\tilde{E}_m, \tilde{H}_m) \right\} \]
Coupled mode theory (CMT)

\[ \sim \exp(i\omega t) \] (FD)

\[ \{ \hat{\epsilon}_m; \beta_m, (\bar{E}_m, \bar{H}_m) \} \]

\[ \left( \begin{array}{c} E \\ H \end{array} \right) (x, y, z) \]
Coupled mode theory (CMT)
Coupled mode theory (CMT)

\[ \sim \exp(i \omega t) \text{ (FD)} \]

\[ \hat{\epsilon}_m; \beta_m, (\vec{E}_m, \vec{H}_m) \]

\[ \left( \begin{array}{c} E \\ H \end{array} \right)(x, y, z) \]

(Next: One of many variants of approaches to CMT.)

(Propagation & interaction of basis fields along a common propagation coordinate.)

[D.G. Hall, B.J. Thompson, Selected papers on Coupled-Mode Theory in Guided-Wave Optics, SPIE Milestone series MS 84 (1993)]

(Codirectional coupling (here), versus contradirectional coupling, coupling to radiation modes, nonlinear coupling.)

(Hybrid variant (HCMT): separate lecture.)
**Coupled mode theory (CMT)**

- Investigate a permittivity \( \hat{\epsilon} \), look for fields \( \mathbf{E}, \mathbf{H} \) with
  \[
  \nabla \times \mathbf{E} = -i\omega \mu_0 \mathbf{H}, \quad \nabla \times \mathbf{H} = i\omega \epsilon_0 \hat{\epsilon} \mathbf{E}.
  \]
  (\( \hat{\epsilon}(x, y, z) \), in general.)

- Available: A set of fields \( \{\mathbf{E}_m, \mathbf{H}_m\} \) for permittivities \( \hat{\epsilon}_m = \hat{\epsilon}_m^\dagger \);
  \[
  \nabla \times \mathbf{E}_m = -i\omega \mu_0 \mathbf{H}_m, \quad \nabla \times \mathbf{H}_m = i\omega \epsilon_0 \hat{\epsilon}_m \mathbf{E}_m.
  \]
  (Not necessarily “modes”.)
Coupled mode theory (CMT)

- Investigate a permittivity $\hat{\epsilon}$, look for fields $E, H$ with
  \[ \nabla \times E = -i \omega \mu_0 H, \quad \nabla \times H = i \omega \epsilon_0 \hat{\epsilon} E. \]
  ($\hat{\epsilon}(x, y, z)$, in general.)

- Available: A set of fields $\{E_m, H_m\}$ for permittivities $\hat{\epsilon}_m = \hat{\epsilon}_m^\dagger$;
  \[ \nabla \times E_m = -i \omega \mu_0 H_m, \quad \nabla \times H_m = i \omega \epsilon_0 \hat{\epsilon}_m E_m. \]
  (Not necessarily “modes”.)

- Assume that $(E, H)$ can be well approximated by
  \[
  \begin{pmatrix}
  E \\ H
  \end{pmatrix}(x, y, z) \approx \sum_m C_m(z) \begin{pmatrix}
  E_m \\ H_m
  \end{pmatrix}(x, y, z),
  \]
  $C_m$: unknown amplitudes, common propagation coordinate $z$.

  (Choose $\hat{\epsilon}_m$ as close as possible to $\hat{\epsilon}$.)
Coupled mode theory (CMT)

(Starting point: a “reciprocity identity”.)

\[ \nabla \cdot (H \times E_l^* - E \times H_l^*) = i \omega \epsilon_0 E_l^* \cdot (\hat{\epsilon} - \hat{\epsilon}_l) E. \]

(Insert CMT ansatz for \( E, H \).)

\[ \ldots \]

(\( \iint dx \, dy, \text{ assume } E_m, H_m \to 0 \text{ for } x, y \to \pm \infty. \))

\[ \ldots \]

(Apply identity \( \nabla \cdot (H_m \times E_l^* - E_m \times H_l^*) = i \omega \epsilon_0 E_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l) E. \))

\[ \ldots \]

(\( \iint dx \, dy, \text{ assume } E_m, H_m \to 0 \text{ for } x, y \to \pm \infty. \))

(Manipulate, arrange terms, tidy up.)
Coupled mode theory (CMT)

(Starting point: a “reciprocity identity”.)

\[ \nabla \cdot (H \times E_l^* - E \times H_l^*) = i \omega \epsilon_0 E_l^* \cdot (\hat{\epsilon} - \hat{\epsilon}_l)E. \]

(Insert CMT ansatz for \( E, H \).)

\[ \int \int d \mathbf{x} d \mathbf{y}, \text{ assume } E_m, H_m \to 0 \text{ for } x, y \to \pm \infty. \]

(Apply identity \( \nabla \cdot (H_m \times E_l^* - E_m \times H_l^*) = i \omega \epsilon_0 E_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l)E. \))

\[ \int \int d \mathbf{x} d \mathbf{y}, \text{ assume } E_m, H_m \to 0 \text{ for } x, y \to \pm \infty. \]

(Manipulate, arrange terms, tidy up.)

\[ \sum_m o_{lm} \partial_z C_m = -i \sum_m k_{lm} C_m \quad \forall l, \quad \text{coupled mode equations.} \]

\[ o_{lm} = \frac{1}{4} \int \int (E_l^* \times H_m - H_l^* \times E_m)_z \, d \mathbf{x} \, d \mathbf{y} = (E_l, H_l; E_m, H_m), \]

\[ k_{lm} = \frac{\omega \epsilon_0}{4} \int \int E_l \cdot (\hat{\epsilon} - \hat{\epsilon}_m)E_m \, d \mathbf{x} \, d \mathbf{y}. \]
Coupled mode theory (CMT)  

\( \nabla \cdot (H \times E_l^* - E \times H_l^*) = i \omega \epsilon_0 E_l^* \cdot (\hat{\epsilon} - \hat{\epsilon}_l)E. \)

(Starting point: a “reciprocity identity”.)

\( \int \int dx \, dy, \ \text{assume} \ E_m, H_m \rightarrow 0 \ \text{for} \ x, y \rightarrow \pm \infty. \)

(Insert CMT ansatz for \( E, H \).)

\( \int \int dx \, dy, \ \text{assume} \ E_m, H_m \rightarrow 0 \ \text{for} \ x, y \rightarrow \pm \infty. \)

(Apply identity \( \nabla \cdot (H_m \times E_l^* - E_m \times H_l^*) = i \omega \epsilon_0 E_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l)E. \))

(Manipulate, arrange terms, tidy up.)

\[ \mathcal{O} \partial_z C = -i \mathcal{K} C, \]

\[ \mathcal{O} = (o_{lm}), \ \mathcal{K} = (k_{lm}). \]

\[ o_{lm} = \frac{1}{4} \int \int (E_l^* \times H_m - H_l^* \times E_m)_z \, dx \, dy = (E_l, H_l; E_m, H_m), \]

\[ k_{lm} = \frac{\omega \epsilon_0}{4} \int \int E_l \cdot (\hat{\epsilon} - \hat{\epsilon}_m)E_m \, dx \, dy. \]
Coupled mode theory (CMT)

\( \mathcal{F}(E, H) = \iiint \left\{ H^* \cdot (\nabla \times E) - E^* \cdot (\nabla \times H) \right. \\
+ i \omega \mu_0 H^* \cdot H + i \omega \epsilon_0 E^* \cdot \hat{\epsilon} E \left. \right\} \, dx \, dy \, dz, \)

\( \delta \mathcal{F} = 0 \ \forall \ \delta E, \delta H \quad \nabla \times E = -i \omega \mu_0 H, \quad \nabla \times H = i \omega \epsilon_0 \hat{\epsilon} E. \)

(Restrict \( \mathcal{F} \) to the CMT ansatz for \( E, H \) \( \sim \) \( \mathcal{F}_c(C) \), require \( \delta \mathcal{F}_c = 0 \ \forall \delta C \).)

\( \cdots \)

\( (\nabla \cdot (H_m \times E_i^* - E_m \times H_i^*)) = i \omega \epsilon_0 E_i^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l) E, \ \iint dx \, dy, \ E_m, H_m \to 0 \ for \ x, y \to \pm \infty. \)

(Manipulate, arrange terms, tidy up.)

\[ \mathcal{O} \frac{\partial}{\partial z} C = -i K C, \]  
\[ C = (C_m), \ \mathcal{O} = (o_{lm}), \ K = (k_{lm}). \]

\( o_{lm} = \frac{1}{4} \iint (E_i^* \times H_m - H_i^* \times E_m)_z \, dx \, dy = (E_l, H_l; E_m, H_m), \)

\( k_{lm} = \frac{\omega \epsilon_0}{4} \iint E_l \cdot (\hat{\epsilon} - \hat{\epsilon}_m) E_m \, dx \, dy. \)
Coupled mode equations

\[ O \frac{\partial}{\partial z} C = -iK C, \quad C = (C_m), \quad O = (o_{lm}), \quad K = (k_{lm}). \]

\[ o_{lm} = \frac{1}{4} \iint (E_i^* \times H_m - H_i^* \times E_m)_z \, dx \, dy = (E_i, H_i; E_m, H_m), \]

\[ k_{lm} = \frac{\omega \epsilon_0}{4} \iint E_i \cdot (\hat{\epsilon} - \hat{\epsilon}_m) E_m \, dx \, dy. \]

- A set of coupled ordinary linear differential equations, of first order. (Here.)
- \( o_{lm} \): power coupling coefficients (field overlaps). (No reason to assume \( o_{lm} = \delta_{lm} \), in general.)
- \( k_{lm} \): coupling coefficients.
- \( z \)-dependence of \( \hat{\epsilon}, \hat{\epsilon}_m, E_m, H_m \) \( \leadsto \) \( o_{lm}(z), k_{lm}(z), O(z), K(z) \). (Compare the bend-straight couplers, Lecture H.)

\[ \ldots \text{ to be solved by numerical procedures.} \] (In general.)
CMT for longitudinally homogeneous structures

\[ \partial_z \hat{\epsilon} = 0, \quad \partial_z \hat{\epsilon}_m = 0, \]

basis: guided modes \[ \left( \begin{array}{c} E_m \\ H_m \end{array} \right)(x, y, z) = \left( \begin{array}{c} \bar{E}_m \\ \bar{H}_m \end{array} \right)(x, y) e^{-i \beta_m z}, \]

\[ \left( \begin{array}{c} E \\ H \end{array} \right)(x, y, z) = \sum_m C_m(z) \left( \begin{array}{c} E_m \\ H_m \end{array} \right)(x, y, z) = \sum_m c_m(z) \left( \begin{array}{c} \bar{E}_m \\ \bar{H}_m \end{array} \right)(x, y). \]

(c_m(z) = C_m(z) \exp(-i \beta_m z), rewrite CMT equations for \( c_m(z) \).)

\[ \nabla \cdot (H_m \times E_i^* - E_m \times H_i^*) = i \omega \epsilon_0 E_i^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l) E, \] integrate, rewrite for \( \bar{E}_m, \bar{H}_m \).

(Symmetrize coefficients.)
CMT for longitudinally homogeneous structures

\[ \partial_z \hat{\epsilon} = 0, \ \partial_z \hat{\epsilon}_m = 0, \]

basis: guided modes \( \begin{pmatrix} E_m \\ H_m \end{pmatrix} (x, y, z) = \begin{pmatrix} \bar{E}_m \\ \bar{H}_m \end{pmatrix} (x, y) e^{-i \beta_m z}, \)

\[ \begin{pmatrix} E \\ H \end{pmatrix} (x, y, z) = \sum_m C_m(z) \begin{pmatrix} E_m \\ H_m \end{pmatrix} (x, y, z) = \sum_m c_m(z) \begin{pmatrix} \bar{E}_m \\ \bar{H}_m \end{pmatrix} (x, y). \]

\( (c_m(z) = C_m(z) \exp(-i \beta_m z), \) rewrite CMT equations for \( c_m(z).) \)

\( (\nabla \cdot (H_m \times E_l^* - E_m \times H_l^*)) = i \omega \epsilon_0 E_l^* \cdot (\hat{\epsilon}_m - \hat{\epsilon}_l) E, \) integrate, rewrite for \( \bar{E}_m, \bar{H}_m.) \)

(Symmetrize coefficients.)

\[ \sum_m \sigma_{lm} \partial_z c_m = -i \sum_m (b_{lm} + \kappa_{lm}) c_m \ \forall l, \]

\[ \sigma_{lm} = \frac{1}{4} \int \int (\bar{E}_l^* \times \bar{H}_m - \bar{H}_l^* \times \bar{E}_m)_z dx \ dy = (\bar{E}_l, \bar{H}_l; \bar{E}_m, \bar{H}_m), \]

\[ \kappa_{lm} = \frac{\omega \epsilon_0}{8} \int \int \bar{E}_l \cdot (\delta \hat{\epsilon}_l + \delta \hat{\epsilon}_m) \bar{E}_m \ dx \ dy, \]

\[ b_{lm} = \sigma_{lm} \frac{\beta_l + \beta_m}{2}. \]
CMT for longitudinally homogeneous structures

\[ \partial_z \hat{e} = 0, \quad \partial_z \hat{e}_m = 0, \]

basis: guided modes \( \left( \begin{array}{c} E_m \\ H_m \end{array} \right)(x, y, z) = \left( \begin{array}{c} \bar{E}_m \\ \bar{H}_m \end{array} \right)(x, y) e^{-i \beta_m z}, \)

\[ \left( \begin{array}{c} E \\ H \end{array} \right)(x, y, z) = \sum_m C_m(z) \left( \begin{array}{c} E_m \\ H_m \end{array} \right)(x, y, z) = \sum_m c_m(z) \left( \begin{array}{c} \bar{E}_m \\ \bar{H}_m \end{array} \right)(x, y). \]

\( (c_m(z) = C_m(z) \exp(-i \beta_m z), \) rewrite CMT equations for \( c_m(z).) \)

\( \left( \begin{array}{c} \nabla \cdot (H_m \times E_l^* - E_m \times H_l^*) \end{array} \right) = i \omega \varepsilon_0 E_l^* \cdot (\hat{e}_m - \hat{e}_l)E, \) integrate, rewrite for \( \bar{E}_m, \bar{H}_m. \)

\( \left( \begin{array}{c} \nabla \cdot (H_m \times E_l^* - E_m \times H_l^*) \end{array} \right) = i \omega \varepsilon_0 E_l^* \cdot (\hat{e}_m - \hat{e}_l)H, \)

(Symmetrize coefficients.)

\[ S \partial_z c = -i(B + Q) c, \quad c = (c_m), \quad S = (\sigma_{lm}), \quad B = (b_{lm}), \quad Q = (\kappa_{lm}), \]

\[ \sigma_{lm} = \frac{1}{4} \int \int (\bar{E}_l^* \times \bar{H}_m - \bar{H}_l^* \times \bar{E}_m) \, dx \, dy = (\bar{E}_l, \bar{H}_l; \bar{E}_m, \bar{H}_m), \]

\[ \kappa_{lm} = \frac{\omega \varepsilon_0}{8} \int \int \bar{E}_l \cdot (\delta \hat{e}_l + \delta \hat{e}_m)E_m \, dx \, dy, \quad b_{lm} = \sigma_{lm} \frac{\beta_l + \beta_m}{2}. \]

\[ \delta \hat{e}_m = \hat{e} - \hat{e}_m, \]
Longitudinally constant structures, coupled mode equations

\[ S \partial_z \hat{c} = -i(B + Q)c, \quad c = (c_m), \quad S = (\sigma_{lm}), \quad B = (b_{lm}), \quad Q = (\kappa_{lm}). \]

\[ \sigma_{lm} = \frac{1}{4} \int \int (\overline{E}_l^* \times \overline{H}_m - \overline{H}_l^* \times \overline{E}_m)_z \, dx \, dy = (\overline{E}_l, \overline{H}_l; \overline{E}_m, \overline{H}_m), \]

\[ \kappa_{lm} = \frac{\omega \varepsilon_0}{8} \int \int \overline{E}_l \cdot (\delta \hat{\epsilon}_l + \delta \hat{\epsilon}_m) \overline{E}_m \, dx \, dy, \quad b_{lm} = \sigma_{lm} \frac{\beta_l + \beta_m}{2}. \]

- A set of coupled ordinary linear differential equations, of first order
- \( \sigma_{lm} \): power coupling coefficients (field overlaps).
- \( \kappa_{lm} \): coupling coefficients.
- \( \partial_z \hat{\epsilon} = \partial_z \hat{\epsilon}_m = 0 \) \( \rightsquigarrow \) \( \partial_z \sigma_{lm} = \partial_z b_{lm} = \partial_z \kappa_{lm} = 0. \)

\( \text{(ODEs with constant coefficients.)} \)

\( \text{\ldots quasi-analytical solutions.} \)
Longitudinally constant structures, coupled mode equations

\[ (\partial_z \hat{\epsilon} = \partial_z \hat{\epsilon}_m = 0) \]

\[ \mathbf{S} \partial_z \mathbf{c} = -i (\mathbf{B} + \mathbf{Q}) \mathbf{c}, \quad \mathbf{c} = (c_m), \quad \mathbf{S} = (\sigma_{lm}), \quad \mathbf{B} = (b_{lm}), \quad \mathbf{Q} = (\kappa_{lm}). \]

\[ \sigma_{lm} = \frac{1}{4} \int \int (\bar{E}_l^* \times \bar{H}_m - \bar{H}_l^* \times \bar{E}_m)_z \, dx \, dy = (\bar{E}_l, \bar{H}_l; \bar{E}_m, \bar{H}_m), \]

\[ \kappa_{lm} = \frac{\omega \epsilon_0}{8} \int \int \bar{E}_l \cdot (\delta \hat{\epsilon}_l + \delta \hat{\epsilon}_m) \bar{E}_m \, dx \, dy, \quad b_{lm} = \sigma_{lm} \frac{\beta_l + \beta_m}{2}. \]

- \[ \sigma_{ml}^* = \sigma_{lm}, \quad b_{ml}^* = b_{lm}; \quad \kappa_{ml}^* = \kappa_{lm}, \quad \text{if } \hat{\epsilon}^\dagger = \hat{\epsilon}, \quad \hat{\epsilon}_m^\dagger = \hat{\epsilon}_m, \]

\[ \mathbf{S}^\dagger = \mathbf{S}, \quad \mathbf{B}^\dagger = \mathbf{B}; \quad \mathbf{Q}^\dagger = \mathbf{Q}, \quad \text{if } \hat{\epsilon}^\dagger = \hat{\epsilon}, \quad \hat{\epsilon}_m^\dagger = \hat{\epsilon}_m. \]

- Power: \[ P = (\mathbf{E}, \mathbf{H}; \mathbf{E}, \mathbf{H}) = \sum_{l,m} c_l^*(\bar{E}_l, \bar{H}_l; \bar{E}_m, \bar{H}_m) c_m = \mathbf{c}^* \cdot \mathbf{Sc} \]

\[ \partial_z P = i \mathbf{c}^* \cdot ((\mathbf{B} + \mathbf{Q})^\dagger - (\mathbf{B} + \mathbf{Q})) \mathbf{c}, \quad \partial_z P = 0 \quad \text{for } B^\dagger = B, \quad Q^\dagger = Q. \]

(For lossless waveguides the scheme is power conservative.)
Longitudinally constant structures, formal solution

\[ S \partial_z c = -i(B + Q)c, \quad \partial_z S = \partial_z B = \partial_z Q = 0. \]

Ansatz: \[ c(z) = a e^{-i b z}, \quad a, b \ \text{constants.} \]

\[ (B + Q)a = b Sa, \quad \text{a generalized eigenvalue problem.} \]

(Dimension: number of basis modes included.)

Solutions: \{a, b\},

"supermodes" \[ \left( \begin{array}{c} E \\ H \end{array} \right) (x, y, z) = \left( \sum_m a_m \left( \begin{array}{c} \bar{E}_m \\ \bar{H}_m \end{array} \right) (x, y) \right) e^{-i b z}. \]

(Superpositions of the original mode profiles with constant coefficients.)
(As many supermodes as there are basis modes.)
(Formalism can be continued: power/orthogonality of supermodes . . .)
Longitudinally constant structures, two coupled modes

Two orthogonal coupled modes \((E_1, H_1), (E_2, H_2)\):

- (Example: two modes supported by the same isotropic waveguide \((\hat{\epsilon}_1 = \hat{\epsilon}_2)\); interaction due to small anisotropy \((\hat{\epsilon})\).)

  (Or: non-orthogonality neglected as a further approximation.)

\[
\sigma_{lm} = (\bar{E}_l, \bar{H}_l; \bar{E}_m, \bar{H}_m) = \delta_{lm}P_0.
\]

  (Orthogonal modes, uniform normalization \(P_m = P_0\).)

  (Or: apply inverse of \(S\) to CM equations, continue with redefined expressions for \(\beta_m, \kappa_{lm}\).)

\[
\left( \begin{array}{c} \partial_z c_1 \\ \partial_z c_2 \end{array} \right) = -i \left( \begin{array}{cc} \beta'_1 & \kappa \\ \kappa^* & \beta'_2 \end{array} \right) \left( \begin{array}{c} c_1 \\ c_2 \end{array} \right), \quad \beta' = \beta_l + \kappa_{ll}/P_0, \quad \kappa = \kappa_{12}/P_0.
\]
Longitudinally constant structures, two coupled modes

Two orthogonal coupled modes \((E_1, H_1), (E_2, H_2)\):

(Example: two modes supported by the same isotropic waveguide \((\hat{\epsilon}_1 = \hat{\epsilon}_2)\); interaction due to small anisotropy \((\hat{\epsilon})\).

(Or: non-orthogonality neglected as a further approximation.)

\[
\sigma_{lm} = (\bar{E}_l, \bar{H}_l; \bar{E}_m, \bar{H}_m) = \delta_{lm} P_0.
\]

(Orthogonal modes, uniform normalization \(P_m = P_0\).

(Or: apply inverse of \(S\) to CM equations, continue with redefined expressions for \(\beta_m, \kappa_{lm}\).)

\[
\begin{pmatrix}
\frac{\partial}{\partial z} c_1 \\
\frac{\partial}{\partial z} c_2
\end{pmatrix}
= -i
\begin{pmatrix}
\beta'_1 & \kappa \\
\kappa^* & \beta'_2
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2
\end{pmatrix},
\]

\[
\beta'_l = \beta_l + \frac{\kappa_{ll}}{P_0}, \quad \kappa = \frac{\kappa_{12}}{P_0}.
\]

\[\Delta \beta' = \beta'_1 - \beta'_2, \quad \rho = \sqrt{\left(\frac{\Delta \beta'}{2}\right)^2 + |\kappa|^2}.
\]
**Longitudinally constant structures, two coupled modes**

Two *orthogonal* coupled modes \((E_1, H_1), (E_2, H_2)\):

(Example: two modes supported by the same isotropic waveguide \((\tilde{\epsilon}_1 = \tilde{\epsilon}_2)\); interaction due to small anisotropy \((\tilde{\epsilon})\).)

(Or: non-orthogonality neglected as a further approximation.)

\[ \sigma_{lm} = (\bar{E}_l, \bar{H}_l; \bar{E}_m, \bar{H}_m) = \delta_{lm} P_0. \]

(Orthogonal modes, uniform normalization \(P_m = P_0\).)

(Or: apply inverse of \(S\) to CM equations, continue with redefined expressions for \(\beta_m, \kappa_{lm}\).)

\[
\begin{pmatrix}
\frac{\partial z c_1}{\partial z c_2}
\end{pmatrix} = -i
\begin{pmatrix}
\beta_1' & \kappa \\
\kappa^* & \beta_2'
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2
\end{pmatrix},
\]

\[ \beta_1' = \beta_1 + \kappa_{ll}/P_0, \]

\[ \kappa = \kappa_{12}/P_0. \]

\[ c_{20} = 0 \implies \left| \frac{c_2(z)}{c_1(0)} \right|^2 = \eta_{\text{max}} \sin^2(\rho z), \quad \eta_{\text{max}} = \frac{|\kappa|^2}{|\kappa|^2 + (\Delta \beta'/2)^2}. \]

\[ \eta_{\text{max}} \text{ at } z = L_c \text{ with } \rho L_c = \pi/2, \]

 coupling length

\[ L_c = \frac{\pi}{\sqrt{(\Delta \beta')^2 + 4|\kappa|^2}}, \]

 (Conversion length, half-beat length.)

\[ \eta_{\text{max}} \text{ at } z = L_c \text{ with } \rho L_c = \pi/2, \]

 coupling length

\[ L_c = \frac{\pi}{2|\kappa|}. \]

(Here the *phase-shifted* propagation constants are relevant.)

(Small interaction (small maximum conversion) for out-of-phase modes, i.e. for \(|\Delta \beta'|^2 \gg |\kappa|^2\)).
Longitudinally constant structures, one “coupled” mode

CMT with one basis mode: \[
\begin{bmatrix} E \\ H \end{bmatrix}(x, y, z) = c_1(z) \begin{bmatrix} \bar{E}_1 \\ \bar{H}_1 \end{bmatrix}(x, y)
\]

\[
\partial_z c_1 = -i \frac{b_{11} + \kappa_{11}}{\sigma_{11}} c_1,
\]

\[
b_{11} = \beta_1, \quad \frac{\kappa_{11}}{\sigma_{11}} = \frac{\omega \varepsilon_0}{2 \text{ Re}} \iint (\bar{E}_1^* \cdot (\hat{\varepsilon} - \hat{\varepsilon}_1) \bar{E}_1) \, dx \, dy =: \delta \beta_1,
\]

\[
\partial_z c_1 = -i (\beta_1 + \delta \beta_1) c_1,
\]

\[
c_1(z) = c_1(0) e^{-i (\beta_1 + \delta \beta_1) z}.
\]

Theory of single mode perturbations.
Upcoming

Next lectures:

- Hybrid analytical / numerical coupled mode theory.
- A touch of photonic crystals; a touch of plasmonics.
- Oblique semi-guided waves: 2-D integrated optics.